On Harrod-Okishio's dynamic model: Some formal comments

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Abstract

The paper examines and extends the Harrod-Okishio model in the two directions. First, plurality of the cenario of the model is examined by means of the Gröbner basis, and a micro-basis of self-finance is pointed out. Secondly, an animal spirit investment function is extended to a logistic map which includes the feedback of the past performance. Such an investment function is seen to possess the possibility of fluctuations or chaos.

1 Preface

On business cycles which periodically visit modern economies, many economists have continued their discussion for many years.

Since business cycles have been observed along with economic growth, and economic growth necessitate investment and hence saving, the theory of business cycles has been discussed with a stress on investment function. Models concerning the economic growth and fluctuations can be classified in several groups, from neo-classical to Marxian, and we shall focus on one of them, that is the system which should be called the Harrod-Okishio model.

At the outset, we summarise the framework of the model, and the main proposition of the Harrod-Okishio model. In Section 2, we make clear some formal aspects of the model by means of the Gröbner basis.

We shall examine the model in two points.

The first is the ambigouity of the cenario of the Harrod-Okishio model, which will be dealt with in Section 3.

The second is the extension of the investment function from the standpoint of the feedback in Section 4.

2 The Harrod-Okishio model

2.1 The fundamental model

In this section, we shall give a brief overview to the Harrod-Okishio model. There are several valuations in the Harrod-Okishio model, but in the following we shall describe their typical model.[1]

Let us write:

ℓ : labour input,
σ : productivity of capital,
X : national income,
K : the amount of capital,
π : the rate of profit,

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[1] As for the Harrod-Okishio model dealt with here, refer to Okishio[1976] and Okishio[1988b].

N : employment,
I : investment,
g : the rate of accumulation,
ω : the real wage rate,
δ : the level of utilization of capital.

ℓ and σ are technically given parameters, whilst all the other economic variables depend on time t. We assume that time is a discrete variable. By employing the above introduced symbols, one has the following equations.

At the outset, the equilibrium condition of the market is given by

$$X(t) = \omega(t)N(t) + I(t), \tag{2.1}$$

whilst income distribution is expressed by profit and wages:

$$X(t) = \pi(t)K(t) + \omega(t)N(t). \tag{2.2}$$

The next two describe the technical relationships between production and employment, and production and the level of utilization of capital.

$$N(t) = \ell X(t), \tag{2.3}$$

$$X(t) = \sigma\delta(t)K(t). \tag{2.4}$$

The decision of the accumulation of capital is expressed by

$$g(t) = \frac{I(t)}{K(t)}, \tag{2.5}$$

whereas the accumulation rule is assumed to obey the *animal spirit* of entrepreneuer:

$$g(t+1) = g(t) + \beta(\delta(t) - 1), \tag{2.6}$$

where β stands for the accerelation coefficient of the capitalist: investment of capital will be increased if the level of utilization of capital is greater than the prescribed normal level 1.

Meanwhile, the level of utilization of capital is influenced by the rate of profit:

$$\delta(t) = \Delta(\pi(t)). \tag{2.7}$$

Finally, the capital accumulation is defined by

$$K(t+1) = K(t) + I(t). \tag{2.8}$$

According to Okishio, those equations are meant to describe the dynamics of the capitalist economy based on the market mechanism with the animal-spirit investment function.

2.2 Formal aspects of the model

The Harrod-Okishio model consists of 8 equations in 8 unknowns, as described in the preceding subsection. The above system of simultaneous equations can be looked at as a system of multi-variable polynomials, so that one can apply the algebra of multi-variable polynomials.

From the angle of formality, the system has a unique set of solutions for all unknowns. This can be easily confirmed by evaluating the Gröbner basis of the system.[2]

[2] As for details of the Gröbner basis, refer to Davenport-Siert-Tournier[1986], pp.97-105.

That is, by computing the Gröbner basis of the system of equations, (2.1)–(2.8), with respect to variables $X(t), \omega(t), N(t), I(t), \delta(t), \pi(t), g(t+1), K(t+1)$, one sees that the original system of eight equations are reduced to

$$
\begin{aligned}
&(2.9) & X(t) &= K(t)\Delta(\pi(t))\sigma \\
&(2.10) & \omega(t) &= \frac{\sigma\Delta(\pi(t)) - g(t)}{\ell\sigma\Delta(\pi(t))} \\
&(2.11) & N(t) &= \ell\sigma K(t)\Delta(\pi(t)) \\
&(2.12) & I(t) &= g(t)K(t) \\
&(2.13) & \delta(t) &= \Delta(\pi(t)) \\
&(2.14) & \pi(t) &= g(t) \\
&(2.15) & g(t+1) &= g(t) + \beta(\Delta(\pi(t)) - 1) \\
&(2.16) & K(t+1) &= (1+g(t))K(t)
\end{aligned}
$$

Algebraic preprocessing of the system of equations of the model is just the first step of the mathematical manipulation.[3] Nevertheless, the Gröbner basis of the system of equations gives a typical simplified representation of the system, and the core of the system is included therein. In fact, (2.14) and (2.15) give the reduced system of the model. This means that these two constitute the ultimate set of independent equations: the other variables are computed from $g(t)$, even if we are not aware of the causal determination indicated by each equation in the original setting.

Since the rate of profit should be equal to the rate of growth, the reduced system takes the same form, even if we interchange those two rates, we can say that the following equation is the core of the Harrod-Okishio model:

$$(2.17) \qquad g(t+1) = g(t) + \beta(\Delta(g(t)) - 1).$$

2.3 Determination of variables from the demand side

At the outset, we apply the assignment approach that equations have an orientation in which the right-hand side determines the variable put at the left-hand side of the equality sign.

From equation (2.4), one has the determination of the level of utilization of capital,

$$(2.18) \qquad \delta(t) = \frac{X(t)}{\sigma K(t)}.$$

From equation (2.6), one determines the amount of investment,

$$(2.19) \qquad I(t) = g(t)K(t).$$

Likewise, from equation (2.3), one has

$$(2.20) \qquad N(t) = \ell X(t),$$

whilst from (2.2),

$$(2.21) \qquad \omega(t) = \frac{X(t) - \pi(t)K(t)}{N(t)}.$$

[3] Algebraic preprocessing of the model is carried out by employing computer algebra systems such as REDUCE and Maple. Log files are found in Appendix.

From the above elimination of four variables, the system of equations is reduced to:

$$\begin{aligned}\pi(t) &= g(t),\\ g(t+1) &= g(t)+\beta\left(1-\frac{X(t)}{\sigma K(t)}\right),\\ \Delta(\pi(t)) &= \frac{X(t)}{\sigma K(t)},\\ K(t+1) &= (1+g(t))K(t),\end{aligned}$$

from the third equation of which one has,

$$X(t)=\sigma K(t)\Delta(\pi(t)). \tag{2.22}$$

Thus, we get the reduced simultaneous system of difference equations as:

$$\pi(t) = g(t) \tag{2.23}$$
$$g(t+1) = g(t)+\beta(\Delta(\pi(t))-1) \tag{2.24}$$
$$K(t+1) = (1+g(t))K(t) \tag{2.25}$$

In the above, we eliminated five equations from the original 8 equations, and thus obtained the reduced system of simultaneous difference equations. This reduction means that if equations (2.23)-(2.25) are solved with respect to $K(t), g(t)$ and $\pi(t)$, then $X(t), N(t), \omega(t)$ and $\delta(t)$ can be obtained by the preceding five equations, (2.18)-(2.22).

Observe that the rate of profit in the above should be understood as the desired rate of profit of the entrepreneurs, whilst the rate of capital accumulation should be the planned rate of capital growth, as well.

Meanwhile, it is worth noting that (2.23) is the metamorphosis of (2.1).[4] This fact cannot be derived from the Gröbner basis of the system.

In any event, the core of the above reduction is again

$$g(t+1)=g(t)+\beta(\Delta(g(t))-1).$$

2.4 The kernel of the Harrod-Okishio model

From the above discussions, we see that the Harrod-Okishio model is reduced to:

$$g(t+1)=g(t)+\beta(\Delta(g(t))-1) \tag{2.26}$$

The equilibrium of the system is given by $\delta(\pi)=1$, which amounts to

$$g(t+1)=g(t).$$

The equilibrium growth rate in the Harrod-Okishio model is the so-called warranted growth rate by Harrod:

$$g^*=\sigma(1-\omega\ell). \tag{2.27}$$

Now, the main formal point of the Harrod-Okishio model is to examine fixed points of a map

$$\psi: x \longrightarrow x+\beta(\Delta(x)-1)$$

in a closed interval which contains g^*.

As for the stability of fixed points of 1-dim maps like ψ, the following lemma can be applied:[5]

[4]This procedure can be realized by the pattern matching of REDUCE. As for its log file, refer to Appendix A.2.

[5]As for the lemma, refer to Hofbauer-Sigmond[1988], pp.36-7. Also see Devaney[1988], chapter 1.

Lemma 2.1 *A fixed point x of a* 1-dim *map f is asymptotically stable, if and only if*

$$|f'(x)| < 1. \tag{2.28}$$

From the above, the following Harrod-Okishio's main formal conclusion is easily derived:

Theorem 2.2 (Harrod-Okishio) *Suppose $\Delta'(g(t)) > 0$. Then, the equilibrium growth rate g^* is not asymptotically stable.*

Moreover, since one has then

$$\frac{\partial g(t+1)}{\partial g(t)} > 1, \tag{2.29}$$

fluctuations from the warranted growth rate, once they are created, are divergent.

We then see the following three remarks:

(1) The order of elimination of $X(t), N(t), \omega(t)$ and $\delta(t)$ can be interchanged: it is not important.

(2) The system can be further reduced to the system of $K(t)$ and $g(t)$, or $K(t)$ and $\pi(t)$. In the latter case, π should be interpreted as the demanded rate of profit.

(3) Irrespective of the choice of the strategic variable, either $g(t)$ or $\pi(t)$, the system reveals the same formal behaviour.

So far, so good. In what follows, we shall examine Harrod-Okishio's model from two major angles, that is, the market equilibrium or Say's law, and the feedback of production as opposed to animal spirit.

3 Equilibirum in the Harrod-Okishio model

3.1 Determination of variables—the other way round

The above assignment approach that carries out the 'Keynesian algorithm' from demand to supply gives us a very impressive point—which equation is reduced to which form. This point, however, should not be overestimated.

The above procedure to reach the reduced system of simultaneous difference equations is not the only way to draw the conclusion. If the above procedure to reduce the system from the demand side is possible, then the one to start from supply is also possible.

That is, we have only to read the market equilibrium condition in the reversed direction, so that the resulting equation (2.14) should be read as the determination of the growth rate by the rate of profit. Differences will appear, however, in the meaning of variables. In this case, the profit rate should be understood as the rate of sales profit. [6]

Nevertheless, it is also seen that the essence of the formal cenario will not change, even if we start from supply.

As shown from the Gröbner basis of the system, these relations of determination of variables do not affect the reduced dynamic system of the model. Irrespective of our interpretation of the causal determination—which equation determines which unknown variable or function, the reduced system is fundamentally the same.

In the above, we see that two economic cenarios can accompany the Harrod-Okishio model, so that it is not correct to confine the Harrod-Okishio model to one particular cenario.

Although the same variables are given different meanings, the reduced dynamic system of equations takes the same form. Why is this possible? This possibility relies on the fact that the Harrod-Okishio model includes the equilibrium condition as an indispencible equation.

[6] The easiest way to see this is to start from a given wage rate. Since one variable is given, one should exclude one of 8 equations. Observe, however, that (2.1), (2.2), (2.6) and (2.8) should not be omitted. It is interesting that from the angle of Gröbner basis, (2.1) can be omitted, but one needs that equation in order to get (2.23). Meanwhile, even if equation (2.3) is dropped, the level of employment can be determined within the system; the production function will then cease to be a linear fixed coefficient function.

3.2 Market equilibrium—unification or assignment

By comparing the Gröbner basis and the assignment approach, we see that even a formally complete system of equations as included in the Harrod-Okishio model is not free from plural interpretations of its internal structure of the system. This may make us feel some kind of ambiguity in the casual determination of variables in the model. It is not too much to say that this ambiguity stems from the market equilibirum.[7] Once the equilibrium condition is introduced into the model, the clear indication that demand, say, should determine supply will be lost. This is because the equality sign has dual meanings.

In mathematics, '=' can either indicate assignment or unification. When looked at from the angle of assignment, $A = B$ means that the value of B is given to A as its value, whilst from the angle of unification, it indicates 'if A is equal to B', so that the value of unification is boolean, i.e., true or false.

Once the equality of supply and demand is established, even if in the sense of unification, its equality permits the interpretation of assignment of the both sides of the equality sign. That is, the assignment of demand to supply and the other way round are both possible in its interpretation.[8]

3.3 Micro-basis of self-finance

Market equilibrium (2.1) is reduced to (2.14). The two are equivalent to each other. That is, (2.1)–(2.8) yield (2.14), and conversely (2.14) and (2.2)–(2.8) yield (2.1).

The meaning of (2.14) in the equilibrium context will be explained as follows.

Suppose that entrepreneurs have to finance their investment by themselves, they will find sources in the profit of their commodities. Then, an immediate behaviour of entrepreneurs will be to determine the desired profit rate by the rate of accumulation. This may be called a micro-basis of self-finance for investment.

If, on the other hand, entrepreneurs decide to invest all of their sales profit, then the rate of sales profit determines the amount of accumulation, and hence the rate of capital growth.

Both cases lead to the equality of supply and demand in the good market. Thus, if we start from (2.14), then the cenario will be easier to be understood.

Nevertheless, self-finance of investment may not be the standard case even if we confine our consideration to the real economy alone.

4 Logistic investment behaviour

The Harrod-Okishio model describes the capitalist economy as an autonomous system. The system seems to be, however, unsatisfactory in the two points.

First, the system does not clearly implement a feedback, such as the dependence of the level of utilization of capital on the past performance.

Secondly, in the Harrod-Okishio model the level of utilization of capital is considered to move both increasingly and decreasingly. In other words, the model is valid in a small interval in which δ can move freely. One of Okishio's contributions is the introduction of the level of utilization of capital into the model, but his formulation does not exploit the importance of this level that δ takes a value in a closed interval.

The degree of utilization of capital should be confined to a closed interval, because there is an upper bound for the degree of utilisation of capital as well as a lower.

Moreover, we should introduce the saturation break effect on the level of utilization of capital; as the level of utilization of capital approaches to its upper bound, increments in it are obliged to become smaller. Observe, however, that the upper bound of the level of utilisation is not a physical limit, but a psychological bound that affects the decision of entrepreneurs.

[7] The ambiguity was first noticed by Shiozawa[1979]. He stresses the manner in which the left-hand side and the right-hand side of the equality sign should be looked at.

[8] Most of the equilibrium models including the complete market adjustment may suffer from the same ambiguity.

Now, let us introduce such a feedback function that incorporates the dependence of the level of the utilization of capital on past perfomance and the closedness of the interval of δ.

Let

m_0 : the minimal degree of utilization of capital,

and the maximum level of $\delta(t)$ be 1.[9]

In view of the above, we consider the feedback equation as follows:

$$\delta(t) = \delta(t-1) + a(1-\delta(t-1))(g(t) - g(t-1)). \tag{4.1}$$

Meanwhile, the equation of capital accumulation rate is replaced by:

$$g(t+1) - g(t) = \beta(\delta(t) - m_0). \tag{4.2}$$

Since equation (4.1) can be easily combined with equation (4.2), one has

$$\delta(t) = -a\beta\delta(t-1)^2 + (1 + a\beta(1+m_0))\delta(t-1) - a\beta m_0. \tag{4.3}$$

Equilibria of (4.3) are given by fixed points of the map

$$\varphi: V \ni x \rightarrow -a\beta x^2 + (1 + a\beta(1+m_0))x - a\beta m_0 \in V,$$

where $V = \{x | \epsilon_1 \leq x \leq \epsilon_2\}$, where ϵ_1 and ϵ_2 are roots of $\varphi(x) = 0$, and the equilibria can be displayed by a graph in the $(\delta(t-1), \delta(t))$ plane, as shown by Figure 4. Remark (4.3) represents a kind of the logistic map.

There are two equilibrium points, $\delta(t) = m_0$ and $\delta(t) = 1$, for φ.

In view of Lemma 2.1 the point $\delta(t) = m_0$ is not asymptotically stable, but repelling; the point $\delta(t) = 1$ is an asymptotically stable equilibrium point, if $|1 - a\beta(1-m_0)| < 1$, that is,

$$0 < a\beta(1 - m_0) < 2; \tag{4.4}$$

otherwise, a chaotic movement may occur. This means that capitalist firms tend to attain the maximum level of the utilization of capital, in so far as the rate of capital accumulation is increased in response to increments in the utilization of capital, although there is a possibility that $\delta(t)$ fluctuates around its maximum level, and hence bounced back. A difference between the Harrod-Okishio model and the system described by (4.3) is that the latter indicates a clear ceiling to the accumulation of capital and the possibility of fluctuations of the degree of capital utilization.

Let us briefly make comparative statics analysis of the above experimental extension. It is rather evident that the stability condition (4.4) is related to attitudes of entrepreneurs. If entrepreneurs are aggressive and hence $a\beta$ is comparatively large, then the equilibrium is likely to be unstable. If entrepreneurs are not demanding, so that the minimum acceptible degree of utilization of capital m_0 is relatively low, then the euilibrium is likely to be unstable.

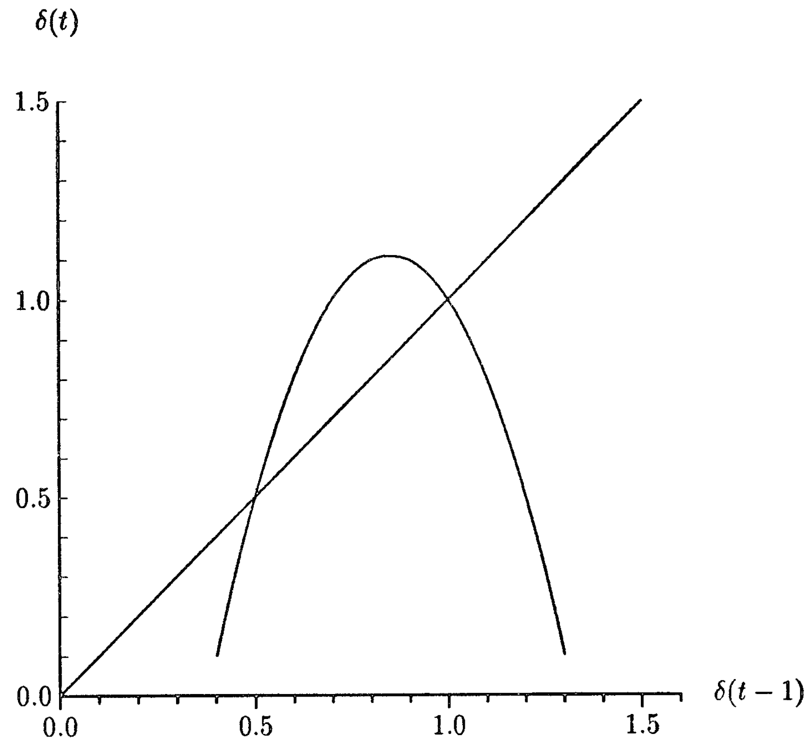
5 Concluding remarks

So far, we examined the Harrod-Okishio model in two points, which are summarized as follows.

First, the Harrod-Okishio model is still an equilibrium model, in which the market equilibrium is focused upon, so that two economic cenarios can accompany the model. The plurality of cenarios may appear as ambiguity of the model, but the formal kernel of the model is not affected.

[9] In the Harrod-Okishio model, $\delta = 1$ is called the normal level of operation, and δ can move around this level without restrictions.

Figure 1: Equilibrium point



Secondly, the Harrod-Okishio type investment function expressing a bare animal spirit explains a process of expansion, but it still remains a short-run or local theory, because the range of the level of utilization of capital is not confined to a closed interval.

By introducing a logistic investment behaviour, we can add, in the framework of the Harrod-Okishio model, the explaination of the process which leads to the apex of boom from depression.

Now, it must be remarked, however, that once the utilization of capital reaches its maximum level, it stopps functioning as a censor for investment, in case the stability condition (4.4) is fulfilled. From the angle of formality, the state of the economy with full capital utilization and a constant growth rate is balanced growth of the economy. The economy never bounces in the setting so far discussed. This suggests that in order to explain crises which may follow booms one should introduce another investment function.

If (4.4) is not met, the economy may fluctuate around the warranted growth path.

Therefore, it will be very difficult to explain crises with only one equation, so that we have to mention that our extension of the Harrod-Okishio model still leaves much to be desired.

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A Formal manipulations by REDUCE

In this appendix, we explain very briefly about the two methods of solving the simultaneous system of algebraic equations. Both reduce the original system to its equivalent simpler system.

Since the computer system cannot accept Greek symbols, we suppose that the variables in the computer session corresponds to the original variable as shown in the following table.

math symbol	REDUCE symbol	indication
X	x	the level of income
K	k	the amount of capital
N	n	employment
I	dk	investment of capital
δ	d	operation level
Δ	delta	reaction function
β	b	reaction coefficient
ℓ	l	labour-input coefficient
σ	s	productivity of capital
t	j	period

A.1 Gröbner basis

The Gröbner basis algorithm in that field is now implemented in REDUCE, and the following shows what happens by that:

```
1: load groebner;
2: clear x,w,n,dk,r,d,g,k,delta;
3: operator x,w,n,dk,r,d,g,k,delta;
4: h:= {x(j) = w(j) * n(j) + dk(j),
      x(j) = r(j) * k(j) + w(j) * n(j),
      n(j) = l * x(j),
      k(j+1) = k(j) + dk(j),
      x(j) = d(j) * sigma * k(j),
      g(j) = dk(j) / k(j),
      d(j) = delta ( r(j)),
      g(j+1) = g(j) + b*(d(j)-1)}$
5: h1 := for each i in h collect num (lhs i - rhs i)$
6: v := {x(j), w(j),n(j),dk(j),d(j),r(j),g(j+1),k(j+1)}$
7: groebner (h1, v);

{X(J) - K(J)*DELTA(R(J))*SIGMA,

        G(J) - DELTA(R(J))*SIGMA
W(J) + --------------------------,
         DELTA(R(J))*L*SIGMA

N(J) - K(J)*DELTA(R(J))*L*SIGMA,
```

```
= DK(J) + G(J)*K(J),

D(J) = DELTA(R(J)),

R(J) = G(J),

G(J + 1) = (G(J) + DELTA(R(J))*B - B),

K(J + 1) = (G(J)*K(J) + K(J))}
```

A.2 Assignment approach by pattern matching

The session on the computer proceeds as follows:[10]

```
1: clear x,w,n,dk,r,d,g,k,delta;
2: operator x,w,n,dk,r,d,g,k,delta;
3: h:= {x(J) = w(j) * n(J) + dk(j),
       x(j) = r(j) * k(j) + w(j) * n(j),
       n(J) = l * x(j),
       x(j) = d(j) * sigma * k(j),
       g(j) = dk(j) / k(j),
       g(j+1) = g(j) + b*(d(j)-1),
       d(j) = delta ( r(j)),
       k(j+1) = k(j) + dk(j)}$
4: h1 := for each i in h collect lhs i - rhs i$
```

where h is a set of basic equations. We transform each equation into the corresponding form whose right-hand side value is zero, the set of which is called h_1.

After declaring the four causal relationships concerning $\delta(t), I(t), N(t)$ and $\omega(t)$ from (2.4), (2.5), (2.3) and (2.2), that is;

```
5: for all j let d(j) = x(j)/(sigma*k(j));
6: for all j let dk(j) = k(j)*g(j);
7: for all j let n(j) = l*x(j);
8: for all j let w(j) = (x(j) - r(j)*k(j))/n(j);
```

and one has:[11]

```
9: h1;

{- K(J)*(G(J) - R(J)),0,0,0,0,

  G(J+1)*K(J)*SIGMA - G(J)*K(J)*SIGMA + K(J)*B*SIGMA - X(J)*B
 ------------------------------------------------------------,
                        K(J)*SIGMA

   K(J)*DELTA(R(J))*SIGMA - X(J)
 - -----------------------------,
            K(J)*SIGMA

 - (G(J)*K(J) - K(J+1) + K(J))}
```

[10] Small typewriter print indicates inputs from the keyboard, while the system response of REDUCE is usually represented by the capital character, but for the sake of clearness, we translate the system output into the ordinary typography.

[11] If one calls h_1 in each step of pattern matching, one can see the metamorphoses of equations.

that is,

$$K(t)(\pi(t)-g(t)) = 0 \tag{A.1}$$

$$g(t+1)-g(t)-\beta\left(1-\frac{X(t)}{\sigma K(t)}\right) = 0 \tag{A.2}$$

$$\Delta(\pi(t))-\frac{X(t)}{\sigma K(t)} = 0 \tag{A.3}$$

$$K(t+1)-(1+g(t))K(t) = 0 \tag{A.4}$$

We can now add one more determination of the equilibrium production level from (A.3) as:

```
10: solve(part(h1,7),x(j))$
11: for each i in h1 collect sub (first ws 10,i);

{- K(J)*(G(J) - R(J)),0,0,0,0,

 G(J+1) - G(J) - DELTA(R(J))*B + B, 0,

 - (G(J)*K(J) - K(J+1) + K(J))}

12: h2 := {first ws, part(ws,6),part(ws,8)}$

13: solve (h2, r(j),g(j+1),k(j+1));

{{R(j)=G(J),

  G(J+1)=G(J)+DELTA(R(J))*B-B,

  K(J+1)=K(J)*(1+G(J))}}
```

Thus, we obtain the kernel of the Harrod-Okishio model.